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Operatori sferico settoriali e semigruppi
in ambiente noncommutativo

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Real and complex operator semigroups

Let Y be a real or complex Banach space.

$\mathsf{T} : [0,\infty[\longrightarrow \mathscr{L}(Y)$ is called *(operator) semigroup in* Y if

$$\begin{cases} \mathsf{T}(t+s) = \mathsf{T}(t)\mathsf{T}(s) \qquad \forall t, s > 0 \\ \mathsf{T}(0) = \mathsf{Id}. \end{cases}$$

T is *uniformly continuous* if: $\mathsf{T} \in C([0,\infty[\,; \mathscr{L}(Y))$.

T is *strongly continuous* if: $\mathsf{T}(\cdot)y \in C([0,\infty[\,; Y) \quad \forall y \in Y$.

Basic example: the exponential

If $\mathsf{A} \in \mathscr{L}(Y)$

$$\mathsf{T}(t) := e^{t\mathsf{A}} = \sum_{n \geq 0} \frac{t^n \mathsf{A}^n}{n!}$$

then T is a uniformly continuous semigroup.

This is the only uniformly continuous semigroup

Theorem.

(a) *If* $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}(Y)$ *unif. continuous semigroup, then*

$$\begin{cases} \mathsf{T}'(t) = \mathsf{A}\mathsf{T}(t) & \forall t \geq 0 \\ \mathsf{T}(0) = \mathsf{Id}, \end{cases}$$

with $\mathsf{A} := \mathsf{T}'(0) \in \mathscr{L}(Y)$ *and* $\mathsf{T}(t) = e^{t\mathsf{A}}$ *for all* $t \geq 0$.

(b) *If* $\mathsf{A} \in \mathscr{L}(Y)$ *then* $\mathsf{T}(t) := e^{t\mathsf{A}}$ *is the unique solution of*

$$\begin{cases} \mathsf{T}'(t) = \mathsf{A}\mathsf{T}(t) & \forall t \geq 0 \\ \mathsf{T}(0) = \mathsf{Id} \end{cases}$$

The generator of a semigroup

If T is *not* unif. continuos, then it is not the exponential series. We need the following notions.

Let $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}(Y)$ be a strongly continuous semigroup. The *generator* of T is the operator $\mathsf{A} : D(\mathsf{A}) \longrightarrow Y$ defined by

$$D(\mathsf{A}) := \left\{ y \in Y \ : \ \exists \lim_{h \to 0} \frac{\mathsf{T}(h)y - y}{h} = \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{T}(t)y \Big|_{t=0} \right\},$$

$$\mathsf{A}y := \lim_{h \to 0} \frac{\mathsf{T}(h)y - y}{h}, \qquad \forall y \in D(\mathsf{A}).$$

Spectral notions

Let $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ be closed and linear.

- *spectrum* of A:

$$\sigma(\mathsf{A}) := \{\lambda \in \mathbb{C} \,:\, \lambda\, \mathsf{Id} - \mathsf{A} \text{ is not bijective}\}.$$

- *resolvent set of* A:

$$\rho(\mathsf{A}) := \mathbb{C} \setminus \sigma(\mathsf{A}).$$

- *resolvent operator of* A *at* α:

$$\mathsf{R}_\lambda(\mathsf{A}) := (\lambda\, \mathsf{Id} - \mathsf{A})^{-1} : X \longrightarrow D(\mathsf{A}), \qquad \lambda \in \rho(\mathsf{A})$$

Generation theorem, part (a)

Theorem (Feller, Miyadera, Phillips (a))**.**

(a) *If* $\mathsf{T} : [0, \infty[\to \mathscr{L}(Y)$ *is a strongly continuous semigroup, then*

$$\exists M \in \mathbb{R}_+,\ \omega \in \mathbb{R}\ :\ \|\mathsf{T}(t)\| \le Me^{t\omega} \quad \forall t \ge 0.$$

and

the generator A *of* T *is closed,* $D(\mathsf{A}) \overset{ds}{\subseteq} Y$*,*

$]\omega, \infty[\ \subseteq \rho(\mathsf{A})$,

$$\|\mathsf{R}_\lambda(\mathsf{A})^n\| \le \frac{M}{(\lambda - \omega)^n} \quad \forall n \in \mathbb{N},\ \forall \lambda > \omega.$$

Generation theorem, part (b)

Theorem (Feller, Miyadera, Phillips (b))**.**

(b) *If* $\mathsf{A} : D(\mathsf{A}) \longrightarrow Y$ *is linear with* $D(\mathsf{A}) \overset{ds}{\subseteq} Y$.
If $\exists\ M \in\ ]1, \infty[,\ \omega \in \mathbb{R}$ *s.t.*

$$]\omega, \infty[\ \subseteq \rho(\mathsf{A}),$$

$$\|\mathsf{R}_\lambda(\mathsf{A})^n\| \leq \frac{M}{(\lambda - \omega)^n} \qquad \forall n \in \mathbb{N}, \quad \forall \lambda > \omega,$$

then A *generates the strongly continuous semigroup*

$$\mathsf{T}(t)y = \lim_{n\to\infty} e^{t\mathsf{A}_n} y, \quad y \in Y,$$

where $\mathsf{A}_n := n\mathsf{A}\mathsf{R}_n(\mathsf{A}) \in \mathscr{L}(Y)$.
Moreover, $\|\mathsf{T}(t)\| \leq Me^{\omega t}$ *for all* $t \geq 0$.

Generation theorem, part (c)

In both cases (a) and (b), we have that

$$\mathsf{R}_\lambda(\mathsf{A})y = \int_0^\infty e^{-t\lambda}\mathsf{T}(t)y\,\mathrm{d}t \qquad \forall\lambda > \omega, \quad \forall y \in Y.$$

Connection to ODE's

Theorem. *If* $\mathsf{A} : D(\mathsf{A}) \longrightarrow Y$ *is closed, linear,* $D(\mathsf{A}) \overset{ds}{\subseteq} Y$*, then the following conditions are equivalent:*

(i) A *generates a strongly continuous semigroup* T*.*

(ii) $\rho(\mathsf{A}) \cap \mathbb{R} \neq \varnothing$,
$\forall y \in D(\mathsf{A})\ \exists! u \in C^1([0,\infty[\,;Y)$ *s.t.* $u(t) \in D(\mathsf{A}) \quad \forall t \geq 0$
and

$$\begin{cases} u'(t) = \mathsf{A}u(t) & \forall t \geq 0, \\ u(0) = y, \end{cases}$$

And we have $u(t) = \mathsf{T}(t)y$*.*

Example: diffusion semigroup

If $Y = L^2(\mathbb{R}^d)$ then

$$\mathsf{T}(t)(f)(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y)\,\mathrm{d}y$$

is a strongly continuous semigroup. Its generator is

$$\mathsf{A} = \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} : W^{1,2}(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$$

thus

$$\begin{cases} \mathsf{T}'(t)y = \mathsf{A}\mathsf{T}(t)y \\ \mathsf{T}(0)y = y \in D(\mathsf{A}) \end{cases} \iff \begin{cases} u'(t) = \Delta u(t) \\ u(0) = y \in W^{1,2} \end{cases}$$

Exponential ?

If A is not continuous, it generates a semigroup $\mathsf{T}(t)$ which is not given by the exponential:

$$\sum_{n\geq 0} \frac{t^n \mathsf{A}^n}{n!} \text{ does not converge.}$$

However it is given by a Cauchy integral formula if A is a *sectorial operator*, i.e. if $\exists \delta \in \,]0, \pi/2[$ s.t.

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{ \lambda \in \mathbb{C} \setminus \{0\} \,:\, |\arg(\lambda)| < \pi/2 + \delta \right\} \subseteq \rho(\mathsf{A}).$$

Operatorial Cauchy integral formula

Theorem. *If* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *is a sectorial operator,*
$\exists M > 0$ *s.t.* $\|\mathsf{R}_\alpha(\mathsf{A})\| \leq M/|\lambda| \qquad \forall\lambda \in \Sigma_\delta$*,*
then

$$\begin{aligned} \mathsf{T}(0) &:= \mathsf{Id}, \\ \mathsf{T}(t) &:= \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}\mathsf{R}_\lambda(\mathsf{A})\,\mathrm{d}\lambda, \qquad t > 0, \end{aligned}$$

is the semigroup generated by A*,*

Γ *being a piecewise* C^1 *curve in* $\rho(\mathsf{A})$
going from $\infty e^{-i(\pi/2+\delta')}$ *to* $\infty e^{i(\pi/2+\delta')}$, $0 < \delta' < \delta$.

Noncommutative setting: Real alternative *-algebras

In general we assume that

- $\mathbb{A}$ is finite dimensional real algebra with unity,
- $\mathbb{A}$ is alternative: $(\alpha, \beta, \gamma) \longmapsto (\alpha\beta)\gamma - \alpha(\beta\gamma)$ alternating.
- $\exists$ *-involution $\mathbb{A} \longrightarrow \mathbb{A} : \alpha \longmapsto \alpha^c$,

  i.e. a real linear mapping such that

$$(\alpha^c)^c = \alpha \qquad \forall \alpha \in \mathbb{A},$$

$$(\alpha\beta)^c = \beta^c \alpha^c \qquad \forall \alpha, \beta \in \mathbb{A},$$

$$\alpha^c = \alpha \qquad \forall \alpha \in \mathbb{R}.$$

Particular cases: $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{R}_{n,m}$

Slice complex nature of $\mathbb{A}$

For any $\mathbf{j} \in \mathbb{A}$ s.t. $\mathbf{j}^2 = -1$ and $\mathbf{j}^c = -\mathbf{j}$ we set:

$$\mathbb{C}_{\mathbf{j}} := \{r + s\mathbf{j} \ : \ r, s \in \mathbb{R}\}$$

and

$$Q_{\mathbb{A}} := \bigcup_{\substack{\mathbf{j}^2=-1 \\ \mathbf{j}^c=-\mathbf{j}}} \mathbb{C}_{\mathbf{j}}.$$

We endow $\mathbb{A}$ with a norm $|\cdot|$ s.t.

$$|\alpha|^2 = \alpha\alpha^c \quad \forall \alpha \in Q_{\mathbb{A}}.$$

Left $\mathbb{A}$-modules

An abelian group $(X, +)$ is a *left* $\mathbb{A}$*-module* if there exists $\mathbb{A} \times X \longrightarrow X : (\alpha, x) \longmapsto \alpha x$, s.t.

$$\alpha(x + y) = \alpha x + \alpha y, \qquad \forall x, y \in X, \quad \forall \alpha \in \mathbb{A},$$

$$(\alpha + \beta)x = \alpha x + \beta x, \qquad \forall x \in X, \quad \forall \alpha, \beta \in \mathbb{A},$$

$$1x = x, \qquad \forall x \in X,$$

$$r(sx) = (rs)x \qquad \forall x \in X, \quad \forall r, s \in \mathbb{R},$$

and, if $\mathbb{A}$ is associative,

$$\alpha(\beta x) = (\alpha\beta)x, \qquad \forall x \in X, \quad \forall \alpha, \beta \in \mathbb{A}.$$

Right $\mathbb{A}$-modules

An abelian group $(X, +)$ is a *right* $\mathbb{A}$*-module* if there exists $X \times \mathbb{A} \longrightarrow X : (x, \alpha) \longmapsto x\alpha$ s.t.

$$(x + y)\alpha = x\alpha + y\alpha, \qquad \forall x, y \in X, \quad \forall \alpha \in \mathbb{A},$$

$$x(\alpha + \beta) = x\alpha + x\beta, \qquad \forall x \in X, \quad \forall \alpha, \beta \in \mathbb{A},$$

$$x1 = x, \qquad \forall x \in X,$$

$$(xr)s = x(rs) \qquad \forall x \in X, \quad \forall r, s \in \mathbb{R},$$

and, if $\mathbb{A}$ is associative,

$$(x\alpha)\beta = x(\alpha\beta), \qquad \forall x \in X, \quad \forall \alpha, \beta \in \mathbb{A}.$$

$\mathbb{A}$-bimodules

An abelian group $(X, +)$ is a $\mathbb{A}$-*bimodule* if:

$(X, +)$ is a left $\mathbb{A}$-module,

$(X, +)$ is a right $\mathbb{A}$-module,

$$rx = xr \qquad \forall x \in X, \quad \forall r \in \mathbb{R},$$

and, if $\mathbb{A}$ is associative,

$$\alpha(x\beta) = (\alpha x)\beta \qquad \forall x \in X, \quad \forall \alpha, \beta \in \mathbb{A}.$$

A useful notation

If X is an $\mathbb{A}$-bimodule, then

$_{\mathbb{A}}X$ means that X is considered as a left $\mathbb{A}$-module,

$X_{\mathbb{A}}$ means that X is considered as a right $\mathbb{A}$-module.

Banach $\mathbb{A}$-bimodule

An $\mathbb{A}$-bimodule X is a *Banach* $\mathbb{A}$-*bimodule* if:
$\exists \|\cdot\| : X \longrightarrow \mathbb{R}_+$ s.t.

$$\|x\| = 0 \iff x = 0,$$

$$\|x + y\| \le \|x\| + \|y\| \qquad \forall x, y \in X,$$

$$\|\alpha x\| \le |\alpha|\,\|x\| \text{ and } \|x\alpha\| \le |\alpha|\,\|x\| \qquad \forall x \in X, \quad \forall \alpha \in \mathbb{A},$$

$$\|\alpha x\| = \|x\alpha\| = |\alpha|\,\|x\| \qquad \forall x \in X, \quad \forall \alpha \in Q_{\mathbb{A}},$$

and if this norm is *complete* in the usual sense.

Right linear operators

Let X be a Banach $\mathbb{A}$-bimodule.

$\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ is a *right linear operator* if

$$D(\mathsf{A}) \text{ is a right } \mathbb{A}\text{-submodule of } X,$$

$$\mathsf{A} \text{ is additive},$$

$$\mathsf{A}(x\alpha) = \mathsf{A}(x)\alpha, \qquad \forall x \in X, \alpha \in \mathbb{A}.$$

$$\mathscr{L}^{\mathrm{r}}(X) := \left\{ \mathsf{A} : X \longrightarrow X \; : \; \mathsf{A} \text{ right linear, } \sup_{x \neq 0} \|\mathsf{A}x\| / \|x\| < \infty \right\}$$
$$= \left\{ \mathsf{A} \in \mathscr{L}({}_{\mathbb{R}}X) \; : \; \mathsf{A} \text{ is right linear} \right\}.$$

$$(\mathsf{A}\alpha)(x) := \mathsf{A}(\alpha x)$$

Operator semigroups in the noncommutative case

Let X be a Banach $\mathbb{A}$-bimodule. $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^{\mathrm{r}}(X)$ is called *(operator) semigroup in X* if

$$\begin{cases} \mathsf{T}(t+s) = \mathsf{T}(t)\mathsf{T}(s) & \forall t, s > 0 \\ \mathsf{T}(0) = \mathsf{Id}. \end{cases}$$

T is *uniformly continuous* if: $\mathsf{T} \in C([0, \infty[\,; \mathscr{L}^{\mathrm{r}}(X))$.
T is *strongly continuous* if: $\mathsf{T}(\cdot)y \in C([0, \infty[\,; X) \quad \forall x \in X$.

Noncommutative spectral notions

Recall the "noncommutative" Cauchy kernel:

$$C_\alpha(\beta) := (\beta^2 - 2\,\mathrm{Re}(\alpha)\beta + |\alpha|^2)^{-1}(\alpha^c - \beta).$$

Idea: replace the resovent operator $\mathsf{R}_\alpha = (\alpha I - \mathsf{A})^{-1}$ by

$$\mathsf{C}_\alpha(\mathsf{A}) = (\mathsf{A}^2 - 2\,\mathrm{Re}(\alpha)\mathsf{A} + |\alpha|^2\,\mathsf{Id})^{-1}(\alpha^c\,\mathsf{Id} - \mathsf{A})$$

F. Colombo, I. Sabadini, D.C. Struppa, *JFA* 2008.

F. Colombo, G. Gentili, I. Sabadini, D.C. Struppa, *JGP* 2010.

Noncommutative spectral notions

$$\Delta_\alpha(\mathsf{A}) : D(\mathsf{A}^2) \longrightarrow X : \Delta_\alpha(\mathsf{A}) := \mathsf{A}^2 - 2\,\mathrm{Re}(\alpha)\mathsf{A} + |\alpha|^2\,\mathsf{Id}.$$

$$\rho_s(\mathsf{A}) := \{\alpha \in Q_{\mathbb{A}} \ : \ \exists \Delta_\alpha(\mathsf{A})^{-1} \in \mathscr{L}^r(X)\},$$

$$\sigma_s(\mathsf{A}) := Q_{\mathbb{A}} \setminus \rho_s(\mathsf{A}),$$

$$\mathsf{C}_\alpha(\mathsf{A}) := \Delta_\alpha(\mathsf{A})^{-1}\alpha^c - \mathsf{A}\Delta_\alpha(\mathsf{A})^{-1}.$$

Remarks:

$$\begin{aligned}\mathsf{C}_\alpha(\mathsf{A}) &= \Delta_\alpha(\mathsf{A})^{-1}\alpha^c - \Delta_\alpha(\mathsf{A})^{-1}\mathsf{A} \\ &= \Delta_\alpha(\mathsf{A})^{-1}(\alpha^c\,\mathsf{Id} - \mathsf{A}) \quad \text{on } D(\mathsf{A});\end{aligned}$$

$$\mathsf{C}_\lambda(\mathsf{A}) = \mathsf{R}_\lambda(\mathsf{A}) \qquad \forall \lambda \in \rho_s(\mathsf{A}) \cap \mathbb{R}.$$

Quaternionic generation theorems

The generation theorems for semigroups in a $\mathbb{H}$-bimodule:

F. Colombo, I. Sabadini, *Adv. Math.* 2011

Tool: quaternionic functional calculus.

A simple key lemma

Lemma. *Let X be a Banach $\mathbb{A}$-bimodule.*
$\mathscr{L}^{\mathrm{r}}(X)$ is closed in $\mathscr{L}({}_{\mathbb{R}}X)$ with respect to the topology of pointwise convergence.

Proof. Let $\mathsf{A}_n \in \mathscr{L}^{\mathrm{r}}(X)$, $\mathsf{A} \in \mathscr{L}({}_{\mathbb{R}}X)$ such that

$$\mathsf{A}_n y \to \mathsf{A} y \qquad \forall y \in X.$$

Then for every $x \in X$, $\alpha \in \mathbb{A}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \|\mathsf{A}(x\alpha) - \mathsf{A}(x)\alpha\| &\leq \|\mathsf{A}(x\alpha) - \mathsf{A}_n(x\alpha) + \mathsf{A}_n(x\alpha) - \mathsf{A}(x)\alpha\| \\ &\leq \|\mathsf{A}(x\alpha) - \mathsf{A}_n(x\alpha)\| + \|\mathsf{A}_n x - \mathsf{A}x\| |\alpha| \end{aligned}$$

Letting $n \to \infty$, we get the right linearity of A. $\square$

The exponential

Theorem (R. Ghiloni, V. Recupero, *Trans. AMS*, in press)**.**

(a) *If* $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^{\mathrm{r}}(X)$ *unif. continuous semigroup, then*

$$\begin{aligned} \mathsf{T}'(t) &= \mathsf{A}\mathsf{T}(t) \qquad \forall t \geq 0, \\ \mathsf{T}(0) &= \mathsf{Id}, \end{aligned}$$

with $\mathsf{A} := \mathsf{T}'(0) \in \mathscr{L}^{\mathrm{r}}(X)$ *and* $\mathsf{T}(t) = e^{t\mathsf{A}}$ *for all* $t \geq 0$*.*

(b) *If* $\mathsf{A} \in \mathscr{L}^{\mathrm{r}}(X)$ *the unif. continuous* $\mathsf{T}(t) := e^{t\mathsf{A}}$ *is the unique solution of*

$$\begin{aligned} \mathsf{T}'(t) &= \mathsf{A}\mathsf{T}(t) \qquad \forall t \geq 0, \\ \mathsf{T}(0) &= \mathsf{Id}. \end{aligned}$$

Proof

(a) If $\mathsf{T} \in C([0,\infty[\,;\mathscr{L}^r(X)) \implies \mathsf{T} \in C([0,\infty[\,;\mathscr{L}({}_{\mathbb{R}}X))$ thus, by the classical theorem,

$$\mathsf{T}'(t) = \mathsf{A}\mathsf{T}(t) \qquad \forall t \geq 0,$$
$$\mathsf{T}(0) = \mathsf{Id},$$

with $\mathsf{A} := \mathsf{T}'(0) \in \mathscr{L}^r(X)$ and $\mathsf{T}(t) = e^{t\mathsf{A}}$ for all $t \geq 0$.

(b) $\mathsf{A} \in \mathscr{L}^r(X) \subseteq \mathscr{L}({}_{\mathbb{R}}X)$ then $\mathsf{T}(t) := e^{t\mathsf{A}}$ is the unique solution of

$$\mathsf{T}'(t) = \mathsf{A}\mathsf{T}(t) \qquad \forall t \geq 0,$$
$$\mathsf{T}(0) = \mathsf{Id}$$

and $\mathsf{T}(t) = e^{t\mathsf{A}} = \sum_n t^n \mathsf{A}^n / n! \in \mathscr{L}^r(X)$.

The generator of a semigroup

The generator is defined as in the real or complex case.

For $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^r(X)$ strongly continuous semigroup, the *generator* of T is the operator $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ defined by

$$D(\mathsf{A}) := \left\{ x \in X \ : \ \exists \lim_{h \to 0} \frac{\mathsf{T}(h)x - x}{h} = \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{T}(t)x\Big|_{t=0} \right\},$$

$$\mathsf{A}x := \lim_{h \to 0} \frac{\mathsf{T}(h)x - x}{h}, \qquad \forall x \in D(\mathsf{A}).$$

Generation theorem in $\mathbb{A}$, part (a)

Theorem (R. Ghiloni, V. Recupero, *Trans. AMS*, in press)**.**

(a) *If* $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^{\mathrm{r}}(X)$ *is a strongly cont. semigroup, then*

$$\exists M \in \mathbb{R}_+, \omega \in \mathbb{R} \ : \ \|\mathsf{T}(t)\| \leq Me^{t\omega} \quad \forall t \geq 0$$

and

the generator A *of* T *is closed,* $D(\mathsf{A})$ *is dense,*

$]w, \infty[\ \subseteq \rho(A)$,

$$\|\mathsf{C}_\lambda(\mathsf{A})^n\| \leq \frac{M}{(\lambda - \omega)^n} \ \forall \ n \in \mathbb{N} \text{ and } \forall \ \lambda > \omega.$$

Generation theorem in $\mathbb{A}$, part (b)

Theorem (R. Ghiloni, V. Recupero, *Trans. AMS*, in press)**.**

(b) *Let* $\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *be right linear s.t.* $D(\mathsf{A}) \overset{ds}{\subseteq} X$*.*
If $\exists\ M \in \mathbb{R}_+,\ \omega \in \mathbb{R}$ *s.t.*

$$]\omega, \infty[\ \subseteq \rho(\mathsf{A}),$$

$$\|\mathsf{R}_\lambda(\mathsf{A})^n\| \leq \frac{M}{(\lambda - \omega)^n} \qquad \forall n \in \mathbb{N}, \quad \forall \lambda > \omega,$$

then A *generates the strongly continuous semigroup*

$$\mathsf{T}(t)y = \lim_{n \to \infty} e^{t\mathsf{A}_n} y, \quad y \in X,$$

where $\mathsf{A}_n := n\mathsf{A}\mathsf{R}_n(\mathsf{A}) \in \mathscr{L}^{\mathrm{r}}(X)$.
Moreover, $\|\mathsf{T}(t)\| \leq Me^{\omega t}$ *for all* $t \geq 0$.

Spherical sectorial operators

$D(\mathsf{A}) \overset{ds}{\subseteq} X$ right $\mathbb{A}$-submodule,
$\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ closed right linear.

A is a *spherical sectorial operator* if $\exists \delta \in \,]0, \pi/2[$ s.t.

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{ \alpha \in Q_{\mathbb{A}} \setminus \{0\} \,:\, \arg(\alpha) < \pi/2 + \delta \right\} \subseteq \rho_s(\mathsf{A}).$$

If A is sectorial and $\exists M > 0$ s.t. $\|\mathsf{R}_\alpha(\mathsf{A})\| \leq M/|\alpha|$ $\forall \alpha \in \Sigma_{\frac{\pi}{2}+\delta}$ is

$$\mathsf{T}(0) := \mathsf{Id},$$
$$\mathsf{T}(t) := \frac{1}{2\pi} \int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A}) \mathbf{j}^{-1} e^{t\alpha} \, \mathrm{d}\alpha, \qquad t > 0,$$

is the semigroup generated by A? The classical proof does not work. Let us differentiate $\mathsf{T}(t)$. We need $\mathbb{A}$ associative.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{T}(t)x &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}e^{t\alpha}\,\mathrm{d}\alpha\ x\right)\\
&= \frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\alpha\ \mathbf{j}^{-1}e^{t\alpha}\,\mathrm{d}\alpha\ x\\
&= \frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} (\mathsf{A}\mathsf{C}_\alpha(\mathsf{A}) + \mathsf{Id})\,\mathbf{j}^{-1}e^{t\alpha}\,\mathrm{d}\alpha\ x\\
&= \frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{A}\mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}e^{t\alpha}\,\mathrm{d}\alpha\ x + \frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathbf{j}^{-1}e^{t\alpha}\,\mathrm{d}\alpha\ x\\
&= \frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{A}\mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}e^{t\alpha}\,\mathrm{d}\alpha\ x\\
&= \mathsf{A}\left(\frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}e^{t\alpha}\,\mathrm{d}\alpha\right)\ x = \mathsf{A}\mathsf{T}(t)x
\end{aligned}
$$

Let us compute $\mathcal{L}(\mathsf{T})(\lambda)$, $\lambda > 0$

$$\begin{aligned}
\int_0^L e^{-t\lambda}\mathsf{T}(t)\,\mathrm{d}t &= \int_0^L e^{-t\lambda}\frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}e^{t\alpha}\,\mathrm{d}\alpha\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}}\int_0^L \mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}e^{t(\alpha-\lambda)}\,\mathrm{d}t\,\mathrm{d}\alpha \\
&= \frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}e^{L(\alpha-\lambda)}(\alpha-\lambda)^{-1} \\
&\quad - \frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}(\alpha-\lambda)^{-1} \\
&\to -\frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}(\alpha-\lambda)^{-1}\,\mathrm{d}\alpha \text{ as } L\to\infty
\end{aligned}$$

$$\mathsf{C}_\lambda(\mathsf{A}) = \mathcal{L}(\mathsf{T})(\lambda), \ \lambda > 0$$

$$\int_0^\infty e^{-t\lambda}\mathsf{T}(t)\,\mathrm{d}t = -\frac{1}{2\pi}\int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\mathbf{j}^{-1}(\alpha-\lambda)^{-1}\,\mathrm{d}\alpha = \mathsf{C}_\lambda(\mathsf{A})$$

Computing the powers of $\mathsf{C}_\lambda(\mathsf{A})$

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}\,\lambda^{n-1}}\mathsf{C}_\lambda(\mathsf{A}) = \frac{\mathrm{d}^{n-1}}{\mathrm{d}\,\lambda^{n-1}}\int_0^\infty e^{-t\lambda}\mathsf{T}(t)\,\mathrm{d}t$$
$$= (-1)^{n-1}\int_0^\infty \mathsf{T}(t)t^{n-1}e^{-t\lambda}\,\mathrm{d}t,$$

but $\mathsf{R}_\lambda(\mathsf{A}) - \mathsf{R}_\mu(\mathsf{A}) = (\mu - \lambda)\mathsf{R}_\lambda(\mathsf{A})\mathsf{R}_\lambda(\mathsf{A})$ yields

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}\,\lambda^{n-1}}\mathsf{C}_\lambda(\mathsf{A}) = (-1)^{n-1}(n-1)!\,(\mathsf{C}_\lambda(\mathsf{A}))^n\,,$$

hence

$$(\mathsf{C}_\lambda(\mathsf{A}))^n = \frac{1}{(n-1)!}\int_0^\infty \mathsf{T}(t)t^{n-1}e^{-t\lambda}\,\mathrm{d}t \qquad \lambda > 0$$

Estimating the powers of $\mathsf{C}_\lambda(\mathsf{A})$

$$(\mathsf{C}_\lambda(\mathsf{A}))^n = \frac{1}{(n-1)!}\int_0^\infty \mathsf{T}(t)t^{n-1}e^{-t\lambda}\,\mathrm{d}t \qquad \lambda > 0$$

gives

$$\begin{aligned}\| \, (\mathsf{C}_\lambda(\mathsf{A}))^n \, \| &\le \frac{1}{(n-1)!}\int_0^\infty C t^{n-1}e^{-t\lambda}\,\mathrm{d}t \\ &= \frac{C}{\lambda^n} \le \frac{C}{(\lambda - r)^n} \qquad \forall\lambda > r\end{aligned}$$

We can apply the generation theorems

A generates a semigroup U and $t \longmapsto \mathsf{U}(t)x$ is the only solution u of

$$\begin{cases} u'(t) = \mathsf{A}u(t) \\ u(0) = x \end{cases}$$

for every $x \in D(\mathsf{A})$. This Cauchy problem is also solved by $\mathsf{T}(t)x$, thus

$$\mathsf{T} = \mathsf{U}$$

and T is a semigroup.

Main theorem

Theorem (R. Ghiloni, V. Recupero, *Trans. AMS*, in press)**.**
If $\mathbb{A}$ *associative,* $\mathbf{j}^2 = -1$, $\mathbf{j}^c = -\mathbf{j}$,
$\mathsf{A} : D(\mathsf{A}) \longrightarrow X$ *is a spherical sectorial operator,*
$\exists M \in \,]0, \infty[$ *s.t.* $\|\mathsf{C}_\alpha(\mathsf{A})\| \leq M/|\alpha| \qquad \forall \alpha \in \Sigma_{\frac{\pi}{2}+\delta}$,
then $\mathsf{T} : [0, \infty[\longrightarrow \mathscr{L}^r(X)$ *defined by*

$$\mathsf{T}(0) := \mathsf{Id},$$

$$\mathsf{T}(t) := \frac{1}{2\pi} \int_{\Gamma_{\mathbf{j}}} \mathsf{C}_\alpha(\mathsf{A})\, \mathbf{j}^{-1} e^{t\alpha} \, \mathrm{d}\alpha, \qquad t > 0,$$

is the semigroup generated by A*.*